

On the Gauss-Bonnet Gravity

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We argue that propagation of gravitational field in the extra dimension is motivated by physical realization of second iteration of self interaction of gravity and it is described by the Gauss-Bonnet term. The most remarkable feature of the Gauss-Bonnet gravity is that at high energy it radically transforms radial dependence from inverse to proportionality as singularity is approached and thereby making it weak. Similar change over also occurs in approach to singularity in loop quantum gravity. It is analogous to Planck's law of radiation where similar change occurs for high and low energy behavior. This is how it seems to anticipate in qualitative terms and in the right sense the quantum gravity effect in 5 dimensions where it is physically non-trivial. The really interesting question is, could this desirable feature be brought down to the 4-dimensional spacetime by dilatonic coupling to the Gauss-Bonnet term or otherwise?

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The most distinguishing feature of gravitation is that it is universal and hence links to everything that physically exists including massive as well as massless particles and above all with itself. Its linkage to massless particles can only be negotiated through curved space [1]. That means gravity must curve spacetime and its dynamics has thus to be entirely determined by the spacetime curvature, the Riemann curvature tensor. We have no freedom to make any prescription, Newton's law should follow from the Riemann curvature. It indeed does through the Bianchi differential identity satisfied by the curvature tensor. It leads to the Einstein equation which contains the Newton's law in the limit [1].

The Einstein equation so obtained naturally contains the so called cosmological constant Λ in a natural way as a constant of integration without any reference to cosmology. It comes on the same footing as the matter tensor in the equation and is indeed a new constant of the theory. It really indicates the distinguishing feature of gravitation that here spacetime background was not fixed as was the case for the rest of physics but was dynamic describing gravitational force. Λ is the measure of this property. The Einstein equation is valid in all dimensions where Riemann curvature is defined, i.e. $n \geq 2$. It is well-known that in dimensions < 4 , it is not possible to realize the free field dynamics. Thus we come to the usual 4-dimensional spacetime. That means 4 dimensions are necessary for description of gravitational field. The question is, are they sufficient too?

Let us now turn to the property of self interaction of gravitation which can be evaluated only by an iteration process [2].¹ The spacetime metric is potential for the Ein-

stein equation which contains its second derivative and square of the first derivative. It thus contains the first order iteration through the square of the first derivative. The natural question that arises is, how do we stop at the first iteration? We should go to second and higher orders as well. The basic entity at our disposal is the Riemann curvature, so should we square it and add to the usual Einstein-Hilbert action of the Ricci scalar? This will also square the second derivative which is the highest order of derivative. If the highest order of derivative does not occur linearly in an equation, then there will be more than one equation, and the question of having unique solution does not arise. The property that highest order of derivative occurs linearly is known as quasi-linearity. Is it then possible to have higher powers of first derivative yet the second derivative remaining linear? Yes, the differential geometry offers a particular combination, known as the Gauss-Bonnet (GB) given by $R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$, which ensures the quasi-linearity character of the resulting equation. This particular combination cancels out the square of the second derivative. However it turns out that this term makes no contribution in the equation for dimension < 5 . We are thus forced to go to the extra 5th dimension for the physical realization of second iteration of self interaction of gravity. This is an important conclusion we have reached simply by hooking onto the iterative realization of self interaction.

Now the question arises, where does this iteration process of going to higher dimension stop? If all the matter fields are confined to 3-brane/space, the 5-dimensional bulk is completely free of matter and hence it is homogeneous and isotropic in space and homogeneous in time, and thereby maximally symmetric. It is therefore of constant curvature, an Einstein space with vanishing Weyl curvature. That means there is no more free gravity to propagate any further in the higher dimension. The iteration chain thus naturally terminates at the second iteration in 5-dimensional bulk for matter fields living on the 3-brane. In how many dimensions should matter live has however to be determined by the dynamics of matter fields.

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¹ There is a long and distinguished history of deriving the Einstein equation from the Newtonian gravity through perturbative inclusion of self interaction (see [3]). However the self interaction we are referring here is the inherent property of the Einsteinian gravity and which is evaluated in the dynamic curved spacetime framework.

The gravitational dynamics in the 5-dimensional bulk [4, 5] is described by

$$G_{AB} = \alpha H_{AB} - \Lambda g_{AB} \quad (1)$$

where $G_{AB} = R_{AB} - \frac{1}{2}Rg_{AB}$, and

$$\begin{aligned} H_{AB} = & -2(RR_{AB} - 2R_{AC}R_B^C - 2R^{CD}R_{ACBD} \\ & + R_A^{CDE}R_{BCDE}) + \frac{1}{2}g_{AB}(R^2 \\ & - 4R_{CD}R^{CD} + R_{CDEF}R^{CDEF}). \end{aligned} \quad (2)$$

Here α is the parameter coupling the Einstein-Hilbert action with the GB term. It is easy to see that the condition of constant curvature solves this equation to give an Einstein space with redefined Λ given by

$$\lambda = \frac{3}{2\alpha}[-1 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda}] \quad (3)$$

which in the first approximation reduces to Λ , $-\Lambda - \frac{3}{\alpha} < 0$ for $\alpha > 0$. The former with +ve sign has the $\alpha \rightarrow 0$ limit leading to the Einstein case. It is flat when $\Lambda = 0$, which means GB contribution has no independent existence. It comes only as correction riding on Λ . In the latter with -ve sign, there is no Einstein limit and the effective λ is always negative leading to AdS. It stands of its own even when Λ vanishes and it can not be switched off. This suggests that its source is not sitting in the bulk. These two cases clearly indicate that they refer to two different situations, and what could they be is what we consider next.

The GB contribution could arise in two different ways. One, when we study the most general action giving rise to quasi-linear equation for gravitation in 5-dimensional spacetime. In this case GB represents the higher order correction and the Einstein gravity results in the $\alpha \rightarrow 0$ limit. This is the case for the +ve sign solution. Second, when GB term is caused by the second order iteration of self interaction of gravitational field whose source is sitting in the 3-brane. It is purely free gravity leaking into the bulk from the brane that sources the GB term in the bulk which can not be switched off in the bulk. This is the case for the -ve sign solution. As argued earlier, bulk spacetime could either be dS or AdS. Since it is free gravity that propagates in bulk and has negative energy density, hence it would generate AdS rather than dS. This demands that the GB parameter α must be positive to give AdS. We thus end up with a scenario similar to the Randall-Sundrum braneworld model (RS) [6] purely from classical consideration without any reference to string theory. Here AdS bulk is not an assumption but follows from the property of gravity. It is therefore no surprise that AdS bulk thus sourced through GB term will also localize gravity on the brane [6, 7], and of course it will have no $\alpha \rightarrow 0$ limit in the bulk. These are the two different situations indicated by the \pm solutions and they

get further resolved when a mass point is introduced. Introduction of a mass point in this setting is described by the well-known Boulware-Deser solution [8, 9] given by

$$ds^2 = -Adt^2 + A^{-1}dr^2 + r^2d\Omega_3^2 \quad (4)$$

where

$$A = 1 - \frac{r^2}{2\alpha}[-1 \pm \sqrt{1 + 4\alpha(\frac{M}{r^4} + \Lambda)}]. \quad (5)$$

Here M is the mass term which has dimension of L^2 , and the two solutions are distinguished by \pm signs. Let us term the +ve sign solution for which the limit $\alpha \rightarrow 0$ exists as the bulk solution (BS) while the -ve sign one has no $\alpha \rightarrow 0$ limit as the brane-bulk solution (BBS). The term under the radical sign must be positive which will be so for $\alpha M > 0, \alpha\Lambda > 0$. For the BS, $M > 0$ and consequently $\alpha > 0$ will accord to the usual attractive gravity in the bulk while it would be repulsive for the BBS case unless we reverse the sign of M, α . Note that the metric is nowhere singular and as $r \rightarrow 0$ it tends to $A \rightarrow 1 - (\pm\sqrt{\frac{M}{\alpha}} + \frac{r^2}{2\alpha})$. In the limit $r = 0$, $A \neq 1$ and hence it is not flat but represents a spacetime asymptotically approximating to a global monopole with a solid angle deficit [10, 11]. The approach to the limit is however through AdS. When $M = 0$, the limiting space is Minkowski flat. Our main aim is to probe GB gravity and hence we shall now set $\Lambda = 0$, which does not play any critical role.

We define the equivalent Newtonian potential, $\Phi = (A - 1)/2$, which leads to gravitational force given by

$$-\Phi' = \frac{r}{2\alpha} \left[-1 \pm \left(1 + 4\alpha \frac{M}{r^4} \right)^{-1/2} \right]. \quad (6)$$

For large r this approximates to the familiar 5-dimensional Schwarzschild for BS while for BBS it is anti-Schwarzschild-AdS unless both M, α are -ve, then it would be Schwarzschild-dS. For smaller r , it goes as $-\frac{r}{2\alpha} \pm O(r^3)$, which shows that approach to the centre $r = 0$ is always through AdS. This demonstrate the remarkable effect of GB contribution which transforms the radial dependence of gravity, from inverse to proportional. This is why the singularity structure is radically altered [12].

The central singularity is however weak because the Kreschmann scalar (square of Riemann curvature) diverges only as r^{-4} . That means energy density will diverge as r^{-2} which on integration over the volume will vanish as $r \rightarrow 0$. This is because at the singularity the metric approximates to that of a global monopole [11] for which this is the characteristic behavior. Thus GB contribution, which would be dominant at high energy as singularity is approached, results in smoothening and weakening the singularity. This is done not by gravity

altering its sense, attraction to repulsion, but by its behavior transforming from inverse square to proportional to r .

The BS solution has the Einstein limit $A = 1 - \frac{m^2}{r^2}$, ($M = m^2$) which is the 5-dimensional Schwarzschild solution. Note that in the first approximation, there is no GB contribution and further the higher order contribution comes as riding on M . It is Minkowski flat when $M = 0$, hence GB contribution has no existence of its own and it comes only as a riding correction. It has horizon at $r_h^2 = m^2 - \alpha$ which will exist only if $m^2 \geq \alpha$, else it will be a naked singularity. Here α behaves like electric charge in the Reissner - Nordström solution for a charged black hole. Its singularity structure would therefore be similar to it. It is quite interesting that asymptotically α has no effect while at the horizon it behaves like a “charge”. Though GB contribution comes in this case only as rider yet its effect becomes dominant as horizon is reached and it radically changes the horizon and singularity structure [12].

The BBS solution with $M = m^2, \alpha > 0$ for large r approximates to $A = 1 + \frac{m^2}{r^2} + \frac{r^2}{\alpha}$, which is AS-AdS. Note that the mass point is repulsive. We could however reverse the situation by taking $M = -m^2, \alpha < 0$, then it would be S-dS. Here the GB contribution comes from gravity leaking from brane into bulk and that produces a spacetime of negative constant curvature, which is AdS. That is why it will always stand of its own and can not be switched off unless one switches off gravity entirely in the brane. Clearly, there is no horizon and there is only weak naked singularity. In this case, the background is set up by gravitational field leaking from the brane into the bulk which should generate an AdS and hence α must be positive. Then addition of a mass point in this setting produces repulsive gravity is the most remarkable and intriguing feature which we do not quite understand.

Our main purpose here was to bring forth and highlight the critical role GB contribution plays. It is however non-trivial only in 5 or higher dimensions. GB gravity arises in two different ways. One, for $n > 4$ dimensions it should be included in the most general action leading to second order quasi-linear equation. It is thus a higher order correction which can not stand all by itself but rides on matter and Λ in the higher dimensional spacetime. On the other hand, GB term could be sourced by free gravity leaking from 3-brane into the bulk as second iteration of self interaction. This stands all by itself and generates an AdS in the bulk. It can not be switched off to give $\alpha \rightarrow 0$ limit simply because its source is not sitting in the bulk but instead in the brane. The bulk is free of matter and hence it is maximally symmetric space of constant curvature which is negative because it is solely produced by free gravitational field having negative energy density. That is why bulk spacetime has to be an AdS and not dS. Note that it is not an assumption but follows from the basic character of gravitational field. On the other hand, in the RS model AdS bulk is required for localization of gravity [6, 7]. Further AdS is also favoured in

a very recent investigation of geodesics and singularities in higher dimensional spacetime [13]. Also note that we have obtained RS model like scenario purely from classical consideration without any reference to string theory. We are driven to the 5-dimensional bulk simply by the physical realization of second order iteration of self interaction of gravity. What the second iteration essentially does is to produce a constant negative curvature in the bulk. A spacetime of constant curvature however solves the equation (1).

The most interesting case is the BBS where there is a gravitational sharing of dynamics between brane and bulk. In this case, there never occurs a horizon irrespective of whether we have the AS-AdS with $M > 0, \alpha > 0$ or S-dS with $M < 0, \alpha < 0$. In the braneworld gravity, AdS bulk is required for localization of gravity on the brane. It has recently been shown that a black hole with sufficiently large horizon on the bulk will delocalize gravity by sucking in zero mass gravitons [14]. A mass point in the GB setting presents a variety of possibilities as there occurs no horizon at all for BBS and even for BS it could be avoided for $m^2 < \alpha$. The absence of horizon altogether in the BBS case is perhaps indicative of the fact that localization of gravity on the brane would continue to remain undisturbed by the introduction of mass point in the bulk. This is perhaps because our interpretation of the BBS is solely guided by the dynamics of gravitational field. In this way, GB could therefore play a very important and interesting role in localizing as well as stabilizing the braneworld gravity [15].

The most distinguishing and characteristic feature of the GB gravity is the negative constant curvature background which manifests as AdS, and its dominance over the mass at high energy as $r \rightarrow 0$ is approached. Asymptotically as $r \rightarrow \infty$, the field goes as r^{-3} for BS and as AdS + r^{-3} for BBS. At the other end, $r \rightarrow 0$, it goes proportional to r . At high energy, gravity effectively changes its radial dependence from inverse to proportionality. This is what is responsible for smoothening and weakening of singularity (Similar indication is also emerging when we consider the dust collapse in the GB setting [16]). This makes the crucial difference in gravitational dynamics at high and low energy. It is something analogous to Planck’s law of radiation which has similarly different behavior at high and low energy. In the loop quantum gravity, apart from gravity turning repulsive there also occurs similar change at high energy as singularity is approached both in cosmology and black hole, density transforms from inverse power to positive power of the scale factor and radius respectively [17, 18, 19]. The GB term, which also arises as one loop contribution in string theory [20, 21], seems to anticipate some aspect of quantum gravity effects at least qualitatively. Thus it could rightly be considered as intermediate limit of quantum gravity. In other way, it could be thought of as right pointer to quantum gravity effects. In the context of loop quantum gravity, we should rather ask for GB gravity as its intermediate limit and so Ads rather than

flat space. That is the limiting continuum spacetime to loop quantum gravity to be rather 5-dimensional AdS than 4-dimensional flat space. This is the suggestion which is naturally emerging and hence should deserve serious further consideration.

Very recently there has been an attempt to see a connection between loop inspired and braneworld cosmology [22]. It is shown that the effective field equations in the two paradigms bear a dynamical correspondence. There appears to be a resonance of it in some other calculation as well [23]. Such a bridge between the two approaches to quantum gravity is quite expected and most desirable as the two refer to complementary aspects. In this perspective, the GB term could also be seen as indicative of a similar bridge between the two approaches. It is quite rooted in the string paradigm through the first loop contribution as well as in the braneworld paradigm. It mimics the features similar to that of the loop quantum calculations at the high energy regime when singularity is approached. Our paradigm makes a very strong suggestion for the intermediate semi-classical limit to the loop quantum gravity as AdS 5-dimensional spacetime rather than 4-dimensional flat spacetime. This is a clear prediction.

There have been several considerations of higher order terms including GB and GB coupled to dilaton in FRW cosmology (see for example [24, 25]). In there, higher order terms act as a matter field in the fixed FRW background simply modifying the Friedman equation. It is a prescription while here we have a true second order quasi-

linear equation to be solved to determine the spacetime metric. The two situations are quite different. The former is an effective modification of the Einstein's theory while the latter is the natural generalization demanded by the dynamics of gravity.

It turns out that GB thus has determining say at high energy. However all this happens in 5 dimensions where GB attains non-trivial physical meaning. It is certainly pointing in the right direction that quantum gravity effects would at the very least weaken singularity if not remove it altogether. The most pertinent question is, could this desirable feature of weakening of singularity be brought down to 4 dimensions through dilaton scalar field coupling to the GB term [26, 27] or otherwise? Very recently, a new black hole solution has been found [28] in which effects of GB and Kaluza-Klein splitting of spacetime manifest in 4 dimensions. What happens is that GB weakens the singularity and regularizes the metric while Kaluza-Klein modes generate the Weyl charge as was the case for one of the first black hole solutions on the Randall-Sundrum brane described by a charged black hole metric [29]. It is remarkable that the new solution asymptotically does indeed approximate to the black hole on the brane.

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